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ON THE MOTION OF A SOLID WITH AN IDEAL NON-RETAINING CONSTRAINT *

A.P. MARKEYEV

The extension of method /1/ of canonical formalism to systems with ideal one-way constraints are applied to the analysis of the motion of a solid when it collides with a stationary horizontal absolutely smooth plane. The surface of the body is assumed to be close to a sphere. The Kolmogorov theorem on the conservation of motion when there is a small change in the Hamiltonian functions /2, 3/ is used for a qualitative investigation of the motion of the body. The existence of periodic motions of an homogeneous ellipsoid of revolution is proved by Poincaré's method /4/ and their stability is investigated.

1. Let a solid with a convex surface with no points and ribs move in a gravitational field above a stationary horizontal surface, which in the course of its motion may collide with the surface. The collision is assumed to be perfectly elastic, and the plane absolutely smooth.

We will relate the motion to a fixed coordinate system $Oxyz$ having its origin at some point of the surface; the Oz axis is directed vertically upward. The system of coordinates $G\xi\eta\zeta$ is rigidly attached to the solid along its principal central axes of inertia. The Euler angles ψ, θ, φ determine the position of the solid relative to the system of coordinates that are conventionally introduced, and the unit vector γ of the Oz axis in the system $G\xi\eta\zeta$ is specified by the components $\gamma_1, \gamma_2, \gamma_3$:

$$\gamma_1 = \sin\theta \sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

Let P be a point of the surface of the solid closest to the horizontal plane Oxy, whose coordinates ξ, η, ζ in the system $G\xi\eta\zeta$ are functions of the angles θ, φ defined by the form of the equations $f(\xi, \eta, \zeta) = 0$ that defines the form of surface of the solid.

If x, y, z are the coordinates of the centre of mass G of the solid in the system $Oxyz$, m is the mass of the body, g is the acceleration due to gravity, A, B, C are the principal central moments of inertia of the solid, p, q, r are the projections of its angular velocity on the axes $G\xi, G\eta, G\zeta$, the kinetic and potential energy of the solid are given by the equations

$$T = {}^1/_2 m(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) + {}^1/_2 (Ap^2 + Bq^2 + Cr^2), \quad \Pi = mgz$$
$$p = \psi^{\cdot}\gamma_1 + \theta^{\cdot}\cos\varphi, \quad q = \psi^{\cdot}\gamma_2 - \theta^{\cdot}\sin\varphi, \quad r = \psi^{\cdot}\cos\theta + \varphi^{\cdot}$$

The non-retaining constraint imposed on the solid is given by the inequality $z \geqslant d$, where $d = -(\xi\gamma_1 + \eta\gamma_2 + \zeta\gamma_3)$ is the distance of the centre of mass from the horizontal plane that passes through the point P. This inequality indicates that the point P is not below the plane Oxy.

Since the two external forces acting on the solid, the gravity force and the force generated on impact, are vertical, the projection of the centre of mass on the Oxy plane moves uniformly and rectilinearly; without loss of generality we will assume that it is stationary ($x^{\cdot} = y^{\cdot} = 0$), so that the centre of mass moves along the given vertical line.

If the coordinate z is replaced by the quantity q_0, using the formula $q_0 = z + \xi\gamma_1 + \eta\gamma_2 + \zeta\gamma_3$, and we put

$$\chi_1 = \xi\sin\varphi + \eta\cos\varphi, \quad \chi_2 = \xi\cos\varphi - \eta\sin\varphi$$

and take into account that the functions $\xi(\theta, \varphi), \eta(\theta, \varphi), \zeta(\theta, \varphi)$ satisfy the easily verified identities

$$\xi'\gamma_1 + \eta'\gamma_2 + \zeta'\gamma_3 \equiv 0$$

where the prime denotes differentiation with respect to θ or φ, the Lagrange function can be expressed in the form

$$L = \frac{1}{2}\sum_{i,j=0}^{3} a_{ij} q_i^{\cdot} q_j^{\cdot} - mg(q_0 - \chi_1 \sin q_1 - \zeta\cos q_1) \tag{1.1}$$

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$$a_{00} = m,\ \ a_{01} = a_{10} = -m\,(\chi_1 \cos q_1 - \zeta \sin q_1),\ \ a_{02} = a_{20} = -m\chi_2 \sin q_1$$

$$a_{03} = a_{30} = 0,\ \ a_{11} = A \cos^2 q_2 + B \sin^2 q_2 + m\,(\chi_1 \cos q_1 - \zeta \sin q_1)^2$$

$$a_{12} = a_{21} = m\chi_2\,(\chi_1 \cos q_1 - \zeta \sin q_1) \sin q_1,\ \ a_{13} = a_{31} = (A - B) \sin q_1 \sin q_2 \cos q_2,\ \ a_{22} = C + m\chi_2^2 \sin^2 q_1$$

$$a_{23} = a_{32} = C \cos q_1,\ \ a_{33} = (A \sin^2 q_2 + B \cos^2 q_2) \sin^2 q_1 + C \cos^2 q_1$$

$$q_1 = \theta,\ \ q_2 = \varphi,\ \ q_3 = \psi$$

The constraint is given by the inequality $q_0 \geqslant 0$.

2. To write the equation of motion in Hamiltonian form it is first necessary to make the non-degenerate change of variables

$$q_0 = Q_0,\ \ q_j = f_j\,(Q_0,\ Q_1,\ Q_2,\ Q_3)\quad (j = 1,\ 2,\ 3) \tag{2.1}$$

selecting the functions f_j so that in the new variables the Lagrange function does not contain products $Q_0^{\cdot} Q_j^{\cdot}$ $(j = 1, 2, 3)$. The functions f_j, according to /1/ must satisfy the system of differential equations

$$a_{1j} \frac{df_1}{dQ_0} + a_{2j} \frac{df_2}{dQ_0} + a_{3j} \frac{df_3}{dQ_0} = -a_{0j}\quad (j = 1, 2, 3) \tag{2.2}$$

Here the quantities q_1, q_2 and q_3 in the coefficients a_{ij} must be replaced by the unknown functions f_1, f_2 and f_3 respectively. The quantity Q_0 in (2.2) plays the part of an independent variable. System (2.2) must be solved for the initial conditions

$$f_j\,|_{Q_0=0} = Q_j\quad (j = 1,\ 2,\ 3) \tag{2.3}$$

The new variables Q_j appear in functions f_1, f_2, f_3 as parameters. We will denote the dynamic system, with Lagrangian function expressed in the new variables Q_0, Q_j $(j = 1, 2, 3)$, by M.

Replacing Q_0 by $|Q_0|$ in the Lagrangian function, we change to the ancilliary system M^* with the Lagrangian function L^*. The trajectories $Q_0(t), Q_j(t)$ and $Q_0^*(t)$. $Q_j^*(t)$ of systems M and M^* satisfy the relations

$$Q_0(t) = |Q_0^*(t)|,\ \ Q_j(t) = Q_j^*(t)\ \ (j = 1,\ 2,\ 3)$$

The ancilliary-system equations may be written in the usual way in the Hamiltonian form

$$\frac{dQ_i}{dt} = \frac{\partial H}{\partial P_i},\quad \frac{dP_i}{dt} = -\frac{\partial H}{\partial Q_i}\quad (i = 0, 1, 2, 3)$$

where we must assume

$$\left.\frac{\partial H}{\partial Q_0}\right|_{Q_0=0} = \min\left(0, \left.\frac{\partial H}{\partial Q_0}\right|_{Q_0=+0}\right)$$

3. The explicit form of the change of variables (2.1) cannot generally be obtained, but if the coefficients a_{0j} $(j = 1, 2, 3)$ are small, the solution of the Cauchy problem (2.2), (2.3) can be found in the form of series in the small parameter. In this problem the coefficients a_{03} is equal to zero, and the coefficients a_{01} and a a_{02} are small if and only if the surface of the solid is close to a sphere whose centre is at the centre of mass of the solid.

To prove this we assume that the equation $f(\xi, \eta, \zeta) = 0$ of the surface of the solid is such that $\gamma = \operatorname{grad} f / |\operatorname{grad} f|$. Then

$$\frac{\partial f}{\partial \xi} = \gamma_1 |\operatorname{grad} f|,\quad \frac{\partial f}{\partial \eta} = \gamma_2 |\operatorname{grad} f|,\quad \frac{\partial f}{\partial \zeta} = \gamma_3 |\operatorname{grad} f|$$

and for the quantities a_{01}, a_{02} we obtain the equations

$$a_{01} = \frac{m}{\sin q_1 |\operatorname{grad} f|^2} \left\{\zeta \left[\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2\right] - \frac{\partial f}{\partial \zeta}\left(\xi \frac{\partial f}{\partial \xi} + \eta \frac{\partial f}{\partial \eta}\right)\right\}$$

$$a_{02} = \frac{m}{|\operatorname{grad} f|} \left(\eta \frac{\partial f}{\partial \xi} - \xi \frac{\partial f}{\partial \eta}\right)$$

The equation $a_{02} = 0$ implies that the solid surface must be a surface of revolution $f(\rho, \zeta) = 0$, where $\rho = (\xi^2 + \eta^2)^{1/2}$. Substituting the function f into the equation $a_{01} = 0$, we obtain

$$\frac{\partial f}{\partial \rho}\left(\zeta \frac{\partial f}{\partial \rho} - \rho \frac{\partial f}{\partial \zeta}\right) = 0 \tag{3.1}$$

The quantity $\partial f / \partial \rho$ cannot be identically zero, since the equation $f = 0$ would then be simply an equation in ζ. Hence from (3.1) it follows that the expression in brackets equals zero, i.e. f is a function of $\rho^2+\zeta^2$. This means that the coefficients a_{01} and a_{02} are identically zero only if the surface of the solid is a sphere whose centre is at the centre of mass G of the solid.

In what follows we will assume the surface of the solid to be close to a sphere of radius R specifedby the equation

$$f \equiv R^2-(\xi^2+\eta^2+\zeta^2)+\mu f_1(\xi,\eta,\zeta,\mu)=0 \quad (0<\mu\ll 1)$$

Function f_1 is assumed to be an analytic function of its variables. The coefficients a_{01} and a_{02} are quantities of the order of μ.

4. The solid is a sphere, when $\mu=0$. The centre of mass of the sphere coincides with its geometric centre; the sphere is generally homogeneous, and has an arbitrary central ellipsoid of inertia. The Lagrangian function L_0 is the sum of two terms

$$L_0=L_0^{(1)}+L_0^{(2)} \tag{4.1}$$

where $L_0^{(1)}$ is the Lagrange function defining the Euler-Poinsot motion of a solid about its centre of mass; it is obtained from the function (1.1), if in the latter we assume $m=0$, $q_0^{\cdot}=0$. The function $L_0^{(2)}$ defines the motion of the centre of mass of the sphere ($q_0=z-R$)

$$L_0^{(2)}={}^1/_2 m q_0^{\cdot 2}-mgq_0, \quad q_0\geqslant 0 \tag{4.2}$$

The subsidiary system M_0^* with the Lagrange function L_0^* has the Hamiltonian function $H_0=H_0^{(1)}+H_0^{(2)}$, where $H_0^{(1)}$ is the Hamiltonian of the Euler-Poinsot problem, and

$$H_0^{(2)}=p_0^2/(2m)+mg\,|q_0| \tag{4.3}$$

The Euler-Poinsot motion is well known. We shall consider in detail the motion of the centre of mass in the system M_0^* which is definedby the Hamiltonian (4.3).

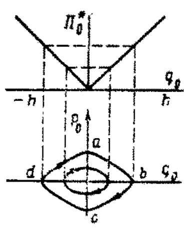

Fig.1

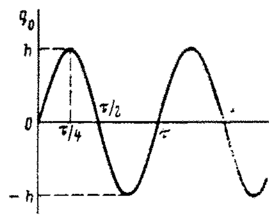

Fig. 2

Curves of the potential energy Π_0^* of the subsidiary system M_0^* and its trajectory in the phase plane q_0, p_0 are shown in Fig. 1. Each phase trajectory consists of parabolic arcs symmetrical about the axis $q_0=0$. In the system M_0^* all motions are periodic of period τ equal to $4\,(2h/g)^{1/2}$, where h is the maximum height the sphere jumps above the plane. The quantity τ is equal to the time interval between the k-th and the $(k+2)$-th collisions of the sphere with the plane.

The dependence of q_0 on time is shown in Fig. 2 for the M_0^* system on the assumption that q_0 is zero at $t=0$. The curve of the function $q_0(t)$ consists of pieces of parabolas

$$q_0(t)=\begin{cases}(2gh)^{1/2}t-gt^2/2, & 0\leqslant t\leqslant \tau/2\\ -(2gh)^{1/2}(t-\tau/2)+g(t-\tau/2)^2/2, & \tau/2\leqslant t\leqslant\tau\end{cases}$$

The actual motion of the centre of mass, described by the Lagrangian function (4.2), has a period of $\tau/2$ equal to the time interval between two consecutive collisions of the sphere with the plane. The curve $q_0(t)$ of the actual motion is obtained from the curve represented in Fig. 2 by a mirror reflection of its part lying below the time axis, relative to that axis. The phase pattern of the actual motion is obtained from the phase pattern in Fig. 1, by taking in the latter only the parts of the trajectory on which $q_0\geqslant 0$, and supplementing them to closed curves by corresponding segments of the $q_0=0$ axis. For instance, to obtain from the phase trajectory $abcd$ of the subsidiary system the phase trajectory of the actual system it is necessary to complement the arc abc by the segment ca. The transfer from point c to point a occurs jumpwise, which corresponds to the collision of the sphere with the plane.

To continue the analysis it is necessary to reduce function (4.3) to action variables - the angle W. For this we make the following 2π periodic in W canonical replacement of the variables q_0, $p_0\to I$, W:

$$q = \begin{cases} 4h\pi^{-2}W(\pi - W), & 0 \leqslant W \leqslant \pi \\ -4h\pi^{-2}(W-\pi)(2\pi - W), & \pi \leqslant W \leqslant 2\pi \end{cases} \tag{4.4}$$
$$p = \begin{cases} 2m\pi^{-1}(2gh)^{1/2}(\pi/2 - W), & 0 \leqslant W \leqslant \pi \\ -2m\pi^{-1}(2gh)^{1/2}(3\pi/2 - W), & \pi \leqslant W \leqslant 2\pi \end{cases}$$

The Hamiltonian (4.3) in variables I, W takes the form

$$H_0^{(2)} = ({}^9/_{32}m\pi^2g^2)^{1/3}I^{2/3} \tag{4.5}$$

The height of the jump is related to the quantity I by the equation

$$I = {}^4/_3 m\pi^{-1}(2g)^{1/2}h^{3/2} \tag{4.6}$$

The frequency ω of the periodic motion of the centre of mass in the subsidiary system $M_0{}^*$ is calculated using the formula

$$\omega = \frac{\partial H_0^{(2)}}{\partial I} = \frac{2}{3}\left(\frac{9m\pi^2g^2}{32I}\right)^{1/3} = \frac{2\pi}{\tau} = \frac{\pi}{2}\left(\frac{g}{2h}\right)^{1/2} \tag{4.7}$$

Hence, for $\mu = 0$ the motion of the subsidiary system is the motion of the Euler-Poinsot solid relative to the centre of the mass and the periodic motion of the latter with frequency ω given by formula (4.7). The actual motion differs from that motion only in the fact that in it $q_0 \geqslant 0$, and the periodic motion of the centre of mass occurs along the vertical line with a frequency 2ω.

5. Now suppose $\mu \neq 0$. We denote by $I_1, I_2, I_3, W_1, W_2, W_3$ the action-angle variables in the Euler-Poinsot problem /5, 6/. Here I_3 is the projection of the angular momentum of the solid relative to the centre of mass on the vertical line, and I_2 is the modulus of the angular momentum. The angle variable W_3 is cyclic in this problem, hence I_3 is the first integral which we shall consider as one of parameters. The Hamiltonian can be written in the form

$$H = H_0(I_1, I_2, I) + \mu H_1(I_1, I_2, I, W_1, W_2, W) + \ldots \tag{5.1}$$
$$H_0 = H_0^{(1)}(I_1, I_2) + H_0^{(2)}(I)$$

where $H_0^{(1)}$ is the Hamiltonian of the Euler-Poinsot motion expressed in terms of the action-angle variables, and the function $H_0^{(2)}(I)$ is defined by formula (4.5). The function $H - H_0$ is analytic for all variables and 2π periodic in W_1, W_2 and W_3. The dots, here and subsequently denote terms of second and higher orders.

Let the motion of the solid relative to the centre of mass be conditionally periodic. Its frequencies ω_k are equal to the derivatives $\partial H_0^{(1)}/\partial I_k$ $(k = 1, 2)$, computed for initial values of the quantities I_1, I_2.

For the Hessian Γ of the function H_0 we have the expression

$$\Gamma = \begin{vmatrix} \dfrac{\partial^2 H_0^{(1)}}{\partial I_1^2} & \dfrac{\partial^2 H_0^{(1)}}{\partial I_1 \partial I_2} & 0 \\ \dfrac{\partial^2 H_0^{(1)}}{\partial I_2 \partial I_1} & \dfrac{\partial^2 H_0^{(1)}}{\partial I_2^2} & 0 \\ 0 & 0 & \dfrac{\partial^2 H_0^{(2)}}{\partial I^2} \end{vmatrix} = \left(\frac{\partial\omega_1}{\partial I_1}\frac{\partial\omega_2}{\partial I_2} - \frac{\partial\omega_1}{\partial I_2}\frac{\partial\omega_2}{\partial I_1}\right)\frac{\partial\omega}{\partial I}$$

The first factor in the above formula for Γ is non-zero, as shown in /7/. Since (4.7) implies that

$$\frac{\partial\omega}{\partial I} = -\frac{4\omega^4}{m\pi^2g^2} \neq 0 \tag{5.2}$$

we have $\Gamma \neq 0$. Hence from the Kolmogorov's theorem /2, 3/ it follows that for any $\varepsilon > 0$ there exists a μ_0 such that for $0 < \mu < \mu_0$ for all t, $-\infty < t < \infty$ motion in the system with Hamiltonian (5.1) conforms to the inequalities $|I_k(t) - I_k(0)| < \varepsilon$ $(k = 1, 2)$, $|I(t) - I(0)| < \varepsilon$ for the majority of initial values $I_k(0), I(0)$ in the sense of Lebesgue measure.

This conclusion means that (for the majority of initial conditions) in an infinite time interval, the motion of the solid of a form fairly close to a sphere is such that its angular momentum and the angle between the angular momentum vector and the vertical differ only little from their initial values; the jump height h above the plane differs little from its value in the unperturbed motion $(\mu = 0)$, and, consequently, the time intervals between two consecutive collisions between solid and plane vary only slightly.

6. Let the solid be dynamically and geometrically symmetrical with axis of symmetry $G\zeta$. The angle of eigenrotation φ is a cyclic coordinate. The projection $L = Cr$ of the vector of angular momentum on the $G\zeta$ axis is the first integral. Having fixed L we relate it to

the parameters of the problem. Investigation of the motion of a solid reduces to a consideration of a system with two degrees of freedom whose Hamiltonian in action-angle variables has the form

$$H = H_0 + \mu K_1 (I_2, I, W_2, W) + \ldots \qquad (6.1)$$
$$H_0 = \frac{I_2^2}{2A} + \left(\frac{9m\pi^2 g^2}{32}\right)^{1/3} I^{2/3}$$

where $H - H_0$ is an analytic function of its variables, 2π periodic in W_2 and W.

The motion of the solid relative to its centre of mass becomes a regular precession when $\mu = 0$. In special cases the precession degenerates into a steady rotation of the solid about its principal central axes of inertia.

The conclusions in Sect. 5 on the conservation of motion for small μ are valid in this case. But, since in this case, the problem reduces to an investigation of systems with two degrees of freedom, not as in Sect. 5 with three degrees, it is possible here to draw conclusions on the conservation of motions for all initial conditions, and not for the majority of cases as in the latter. To check this it is sufficient to verify condition $D \not\equiv 0$ /3/ of the isoenergetic non-degeneration function H_0 in (6.1). We have

$$D = \begin{vmatrix} \frac{\partial^2 H_0}{\partial I_2^2} & \frac{\partial^2 H_0}{\partial I_2 \partial I} & \frac{\partial H_0}{\partial I_2} \\ \frac{\partial^2 H_0}{\partial I \partial I_2} & \frac{\partial^2 H_0}{\partial I^2} & \frac{\partial H_0}{\partial I} \\ \frac{\partial H_0}{\partial I_2} & \frac{\partial H_0}{\partial I} & 0 \end{vmatrix} = -\frac{\omega^2}{A} - \left(\frac{I_2}{A}\right)^2 \frac{\partial \omega}{\partial I}$$

Taking into accoung relations (4.6), (4.7), and (5.2), we obtain

$$D = \frac{\omega^2}{A}\left(\frac{I_2^2}{2mghA} - 1\right)$$

The quantity D vanishes if and only if $mgh = I_2^2/2A$. Hence $D \not\equiv 0$. Because of this the conclusions about the motions of a solid are valid here for all initial conditions.

7. Let us consider the question of the existence and stability of periodic motions of a solid that is nearly a sphere. To be specific we shall consider the solid to be a homogeneous ellipsoid, whose surface equation has the form $1 - (\xi^2/a^2 + \eta^2/b^2 + \zeta^2/c^2) = 0$. The coordinates ξ, η, ζ of point P are related to the Euler angles θ, φ by the relations

$$b^2c^2\xi = -\Delta \sin\theta \sin\varphi, \quad c^2a^2\eta = -\Delta \sin\theta \cos\varphi, \quad a^2b^2\zeta =$$
$$-\Delta \cos\theta$$
$$\Delta = (b^4c^4\xi^2 + c^4a^4\eta^2 + a^4b^4\zeta^2)^{1/2}$$

The following equations hold:

$$\chi_1 = -\frac{\Delta}{a^2b^2c^2} \sin\theta\, (a^2 \sin^2\varphi + b^2\cos^2\varphi),$$
$$\chi_2 = -\frac{\Delta\,(a^2 - b^2)}{a^2b^2c^2} \sin\theta \sin\varphi \cos\varphi$$

$$\chi_1 \cos\theta - \zeta \sin\theta = -\frac{\Delta}{a^2b^2c^2} \sin\theta\cos\theta\, (a^2\sin^2\varphi + b^2\cos^2\varphi - c^2)$$
$$\chi_1 \sin\theta + \zeta\cos\theta = -\frac{\Delta}{a^2b^2c^2}\left[(a^2\sin^2\varphi + b^2\cos^2\varphi)\sin^2\theta + c^2\cos^2\theta\right]$$
$$A = m\,(b^2 + c^2)/5, \quad B = m\,(c^2 + a^2)/5, \quad C = m\,(a^2 + b^2)/5$$

Suppose the semi-axes of the ellipsoid are close to R, differing from it by quantities of order μR. The Lagrange function (1.1) may be represented in the form

$$L = L_0 + m\,(a - b)\sin^2 q_1 \sin 2q_2 q_0^{\cdot} q_2^{\cdot} + m\,[(b - c) +$$
$$(a - b)\sin^2 q_2]\sin 2q_1 q_0^{\cdot} q_1^{\cdot} + \ldots$$

where the function L_0 is defined by formula (4.1). The system of equations (2.2), defining together with conditions (2.3) the change of variables (2.1), is:

$$\frac{df_1}{dQ_0} = \frac{5}{2R^2}\left[(c - b) + (b - a)\sin^2 f_2\right]\sin 2f_1 + \ldots$$
$$\frac{df_2}{dQ_0} + \cos f_1 \frac{df_3}{dQ_0} = \frac{5}{2R^2}(b - a)\sin^2 f_1 \sin 2f_2 + \ldots$$
$$\frac{df_3}{dQ_0} + \cos f_1 \frac{df_2}{dQ_0} = \ldots$$

From this we obtain the explicit form of the change of variables (2.1)

$$q_1 = Q_1 + \frac{5}{2R^2} Q_0 [(c-b) + (b-a)\sin^2 Q_2] \sin 2Q_1 + \ldots \tag{7.1}$$
$$q_2 = Q_2 + \frac{5}{2R^2} Q_0 (b-a) \sin 2Q_2 + \ldots$$
$$q_3 = Q_3 - \frac{5}{2R^2} Q_0 (b-a) \cos Q_1 \sin 2Q_2 + \ldots$$
$$q_0 = Q_0$$

Note that because the coordinate ψ is cyclic, the variable Q_3 is also cyclic in the system with the transformed Lagrange function.

By making the change of variables (7.1) we change to the auxiliary system M^*, by substituting $|Q_0|$ for Q_0 and writing the equations of motion of system M^* in Hamiltonian form. As the variables determining the motion of the solid relative to the centre of mass, we take the Andoyer variables, and for determining the motion of the centre of mass we take the variables I -W, defined in Sect. 4. We assume the ellipsoid to be symmetric $(a = b)$. Expressing the remainder of $(c - a)$ in the form $\mu(c' - a')$, where $0 < \mu \ll 1$, and $(c' - a')$ is of the order of unity, after some transformations, using the relation of the Andoyer variables to the Euler angles and their derivatives /6/, we obtain the Hamiltonian function of system M^* in the form

$$H = H_0 + \mu H_1 + \ldots \tag{7.2}$$

where the function H_0 is given by formula (6.1), and

$$H_1(I_2, I, \varphi_2, W) = 2m(c' - a') \sin\delta_1 \sin\delta_2 I_2^2 A^{-2} |q_0(I, W)| \times \tag{7.3}$$
$$(\sin\delta_1 \sin\delta_2 \cos 2\varphi_2 - \cos\delta_1 \cos\delta_2 \cos\varphi_2) + mg(c' -$$
$$a') \cos^2 Q_1$$
$$\cos Q_1 = \cos\delta_1 \cos\delta_2 - \sin\delta_1 \sin\delta_2 \cos\varphi_2, \quad \cos\delta_1 = I_3/I_2,$$
$$\cos\delta_2 = L/I_2$$

Here I_3 and L are the projections of the angular momentum on the vertical and on the axis of symmetry. These are the first integrals and appear in (7.2) as parameters. The variable I_2 is the angular momentum, and φ_2 its conjugate coordinate /6/. Terms of second and higher orders in μ, which do not appear in (7.2), are functions of I_2, I, φ_2, W, and are 2π periodic in φ_2 and W.

The function $q_0(I, W)$ contained in H_1 is determined by (4.4) and (4.6). The function $|q_0|$ can be represented by the Fourier series

$$|q_0(I, W)| = \frac{g}{4\omega^2}\left(\frac{\pi^2}{3} - 2\sum_{n=1}^{\infty} \frac{\cos 2nW}{n^4}\right) \tag{7.4}$$

When $\mu = 0$ the motion a system with the Hamiltonian function (7.2) is defined by the formulae

$$I_2 = I_{20} = \text{const}, \quad I = I_0 = \text{const} \tag{7.5}$$
$$\varphi_2 = \omega_2 t, \quad W = \omega t + \lambda \ (\lambda = \text{const})$$

It is assumed that when $t = 0$, we have $\varphi_2 = 0$. This does not limit the generality, since the equations of motion do not explicitly contain time. In (7.5) $\omega_2 = I_{20}/A$ and ω is calculated using formula (4.7) with $I = I_0$.

Formulae (7.5) describe the uniform rotation of the solid at an angular velocity ω_2 about the invariant vector of angular momentum. Then the solid periodically collides with the plane, at time intervals equal to $\pi\omega$. The maximum jump height h above the plane is calculated by formula (4.6) in which we must set $I = I_0$.

If the ratio ω_2/ω proves to be a rational number, the motion is periodic with some period $\varkappa$. When μ is non-zero, but fairly small, the existence of periodic motions $\varkappa$ can be established by using Poincare's method (/4/, Sect.42).

To do this it is necessary to calculate the mean value $\langle H_1 \rangle$ of the function H_1 on the unperturbed $\varkappa$-periodic motion (7.5).

$$\langle H_1 \rangle = \frac{1}{\varkappa} \int_0^{\varkappa} H_1(I_{20}, I_0, \omega_2 t, \omega t + \lambda)\, dt$$

The quantity $\langle H_1 \rangle$ is a function of I_{20}, I_0 and λ.

If when $I_2 = I_{20}$, $I = I_0$ the Hessian of the function H_0 is non-zero, and the relations

$$\frac{\partial \langle H_1 \rangle}{\partial \lambda} = 0, \quad \frac{\partial^2 \langle H_1 \rangle}{\partial \lambda^2} \neq 0 \tag{7.6}$$

are satisfied for certain $\lambda = \lambda_*$, then for fairly small μ a periodic motion exists that is analytic in μ that transfers into motion (7.5), when $\mu = 0$ in which $\lambda = \lambda_*$.

According to /1, Sect.?? the two corresponding characteristic indices that correspond to that motion are zero, and the two others $\pm\alpha$ can be expanded in converging series in increasing powers of $\mu^{1/2}$

$$\alpha = \alpha_1\mu^{1/2} + \alpha_2\mu + \alpha_3\mu^{3/2} + \dots$$

and

$$\omega_2^2\alpha_1^2 = \frac{\partial^2\langle H_1\rangle}{\partial\lambda^2}\bigg|_{\lambda=\lambda_*}\left(\omega_2^2\frac{\partial^2 H_0}{\partial I^2} - 2\omega_2\omega\frac{\partial^2 H_0}{\partial I_2\partial I} + \omega^2\frac{\partial^2 H_0}{\partial I_2^2}\right) \tag{7.7}$$

Hence follows the condition of orbital stability of the periodic motion in the first linear approximation for fairly small μ: the right side of Eq.(7.7) must be negative.

The question of the orbital stability of periodic motion in a non-linear formulation can be solved using the following condition* (*See A.A. Saimbattalov. Periodic Poincaré solutions and their stability in the problem of the motion of a solid subjected to the action of gravitational moments. Candidate Dissertation, Moscow. Aviat. Inst, Moscow, 1984.) if the condition of orbital stability is satisfied in the first (linear) approximation, then for the periodic motion to be really orbitally stable it is sufficient that the inequality

$$3\frac{\partial^4\langle H_1\rangle}{\partial\lambda^4}\frac{\partial^2\langle H_1\rangle}{\partial\lambda^2} - 5\left(\frac{\partial^3\langle H_1\rangle}{\partial\lambda^3}\right)^2 \neq 0 \tag{7.8}$$

be satisfied for $\lambda = \lambda_*$.

Calculations based on formulae (7.3-(7.5) show that in this problem, to a first approximation in μ, periodic motions are found for which the frequency ω_2 of the unperturbed motions (7.5) is a multiple of the frequency $\omega: \omega_2 = k\omega$ $(k = 1, 2, 3, \dots)$. In such motions (7.5) the solid rotates about the angular momentum vector by the angle $k\pi$ between two consecutive collisions with the plane.

When $\omega_2 = k\omega$, we have

$$\langle H_1\rangle = mg(c' - a')\left(\cos^2\delta_1\cos^2\delta_2 + \frac{1}{2}\sin^2\delta_1\sin^2\delta_2\right) - \tag{7.9}$$
$$\frac{1}{2}mg(c' - a')\sin^2\delta_1\sin^2\delta_2 \times$$
$$\begin{cases} \cos 2\lambda, & k - \text{odd} \\ (\cos 2\lambda - 4\,\mathrm{ctg}\,\delta_1\,\mathrm{ctg}\,\delta_2\cos\lambda), & k - \text{even} \end{cases}$$

It follows from (7.6) and (7.9) that the unknown periodic motions must be such that for them, when $\mu = 0$, the quantity $\sin\delta_1\sin\delta_2$ to non-zero, i.e. the unperturbed periodic motion must not be rotation about the vertical line or about the axis of symmetry.

We shall briefly present the results of a check of conditions (7.6)-(7.8) in this problem.

Let k be an odd number. Two types of periodic motions then exist:

a) $\lambda_* = 0$ or π. The corresponding periodic motion is orbitally stable if the following inequality is satisfied:

$$(c - a)(1 - \beta) < 0; \quad \beta = 1/40\pi^2R^2h^{-2}k^2 \tag{7.10}$$

b) $\lambda_* = {}^1/_2\pi$ or ${}^3/_2\pi$. The corresponding periodic motion is orbitally stable if

$$(c - a)(1 - \beta) > 0 \tag{7.11}$$

If k is an even number, three types of periodic motions exist:

a) $\lambda_* = 0$ and in unperturbed motion $\mathrm{ctg}\,\delta_1\,\mathrm{ctg}\,\delta_2 \neq 1$. The periodic motion is orbitally stable if $\mathrm{ctg}\,\delta_1\,\mathrm{ctg}\,\delta_2 \neq 4$ and the following inequality is satisfied:

$$(c - a)(1 - \mathrm{ctg}\,\delta_1\mathrm{ctg}\,\delta_2)(1 - \beta) < 0 \tag{7.12}$$

b) $\lambda_* = \pi$ and in the unperturbed motion $\mathrm{ctg}\,\delta_1\,\mathrm{ctg}\,\delta_2 \neq -1$. The corresponding periodic motion is orbitally stable if $\mathrm{ctg}\,\delta_1\mathrm{ctg}\,\delta_2 \neq -4$ and

$$(c - a)(1 + \mathrm{ctg}\,\delta_1\,\mathrm{ctg}\,\delta_2)(1 - \beta) < 0 \tag{7.13}$$

c) $\cos\lambda_* = \mathrm{ctg}\,\delta_1\,\mathrm{ctg}\,\delta_2$ $(|\mathrm{ctg}\delta_1\mathrm{ctg}\,\delta_2| < 1)$. The periodic motion is orbitally stable, if

$$(c - a)(1 - \beta) > 0 \tag{7.14}$$

If inequalities (7.10)-(7.14) are satisfied with opposite signs, the corresponding periodic motions are unstable.

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THE EQUATIONS OF MOTION OF A NON-HOLONOMIC SYSTEM WITH A NON-RETAINING CONSTRAINT*

A.P. IVANOV

The regularity of the equations of motion of a system with a perfect non-retaining constraint $q_1 \geqslant 0$ and with differential constraints is demonstrated. As regards the latter it is assumed that they are imposed either on all motions of the system or only on those for which $q_1 = 0$. The effect of impacts on the stability of permanent rotation of a heavy solid about its axis of symmetry above an absolutely rough surface is investigated. It is shown that the stability of rotation of a solid on the surface can be destabilized by tearing away to an arbitary height, as small as desired.

The possibility of deriving the equations of motion in regular form which defines the motion of a holonomic system with non-retaining constraint in an arbitrary time interval was showing earlier /1/. The advantages of this approach in comparison with the traditional method of "fitting" were demonstrated in /2-4/.

1. Suppose we are given a mechanical system M, defined in the configuration space $q \in R^n$ by generalized forces $\mathbf{Q}$ and the kinetic energy T, which is a quadratic form in $\mathbf{q}^\cdot$. The motion of the system is restrained by a non-retaining constraint $q_1 \geqslant 0$, and by $m < n$ differentiable constraints of the form

$$c_i = a_i\mathbf{q}^\cdot = 0,\ a_i = \mathbf{a}_i(\mathbf{q}, t) \in R^n \quad (i = 1, \ldots, m) \tag{1.1}$$

We shall consider two types of differentiable constraints, assuming that only those motions for which $q_1 = 0$, and for $i = m_1 + 1, \ldots, m$ all motions of system M, obey relations (1.1).

If the coordinate q_1 vanishes when $t = t^*$ an impact occurs on the non-retaining constraint, as well as the differential constraints of the first type. According to Newton's hypothesis that impact (considered absolutely elastic) can be defined by the relations

$$q_1^\cdot(t^* + 0) = -q_1^\cdot(t^* - 0),\quad c_j(t^* + 0) = -c_j(t^* - 0) \tag{1.2}$$
$$(j = 1, \ldots, m_1)$$

We describe the motion free of impacts by the Boltzmann-Hamel Eqs./5/. If the quasicoordinates π are defined by a reversible substitution

$$\pi_i^\cdot = \mathbf{l}_i\mathbf{q}^\cdot,\ \pi_{n-m+j}^\cdot = c_j,\ \mathbf{l}_i(\mathbf{q}, t) \in R^n \tag{1.3}$$
$$(i = 1, \ldots, n-m;\ j = 1, \ldots, m)$$

these equations have in region $q_1 > 0$ the form

$$\frac{d}{dt}\frac{\partial T}{\partial \pi_s^\cdot} - \frac{\partial T}{\partial \pi_s} + \gamma_{sij}\frac{\partial T}{\partial \pi_i^\cdot}\pi_j^\cdot = \Pi_s,\quad \pi_r^\cdot = 0 \tag{1.4}$$
$$(s = 1, \ldots, n - m + m_1;\ r = n - m + m_1 + 1, \ldots, n)$$

where the kinetic energy T is set up taking relations (1.3) into account, Π_s is the generalized force corresponding to the quasicoordinate π_s and the coefficients of non-holonomy γ are determined from the permutational relations

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